# 隨機製程變化下變異性管制圖之設計

# Design of dispersion control charts based on random process change

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## Abstract

This paper develops an algorithm for the optimization design of dispersion control charts. The control chart statistics are based on sample range ($R$) and sample standard deviation ($S$). The design optimizes the sample size, sampling interval, and control limits to minimize the mean number of defective units (MD) produced per out-of-control case. The design aims at reducing the quality cost but only requires limited number of process specifications. It provides the control chart designers with an alternative between the conventional statistical design and economic design for statistical process control. The specific character of this design is that the process shifts is treated as a random variable instead of traditional fixed and known magnitude. A numerical example is presented to illustrate the real application of the MD design of dispersion control charts and its comparison with Shewhart control chart.

**Keywords:** Quality Control、Control Chart、Random Process Change.

## 1. Introduction

Statistical process control techniques have been greatly implemented in industries for improving product quality and saving production costs. As a primary tool among these techniques, control charts are widely used to detect the occurrence of assignable causes such that the necessary correction may be taken before a large quantity of nonconforming units are manufactured.

To employ control charts, three control chart parameters should be specified, i.e., the sample size, the sampling interval and the control limits. A heuristic design where samples of size four or five are taken every hour and three sigma limits are used on control charts is most popular in practice because it is easy to be implemented and understood. However, the resulting chart is not guaranteed to be optimal from economical criteria.

Statistical design of control charts is to select the sample size and the width of control limits that meet the statistical constraints specified for type I and type II errors. Economic design of control charts is to determine the sample size, the width of control limits and sampling interval that minimize the cost or maximize the profit associated with the process environment. Duncan (1956) proposed the first economic model to determine chart parameters of the $\overline{X}$ chart when the process is subject to a single assignable cause. The objective of economic design is to find the optimal settings of chart parameters that minimize the associated cost of operating control charts. Lorenzen and Vance (1986) provided the unified cost model to economic design of control charts and a unification of notation.

This paper develops an algorithm for the optimization design of dispersion control charts. The design optimizes the sample size, sampling interval, and control limits to minimize the mean number of defective units (*MD*) produced per out-of-control case. The design aims at reducing the quality cost but only requires limited number of process specifications. It provides the control chart designers with an alternative between the conventional statistical design and economic design for statistical process control. The specific character of this design is that the process shifts is treated as a random variable instead of traditional fixed and known magnitude. The paper is organized as follows. The dispersion control charts are briefly reviewed in Section 2. Section 3 develops the *MD* design of dispersion chart. A numerical example is presented to illustrate the real application of the *MD* design of dispersion control charts in Section 4.

## 2. Review of dispersion control charts

Some factors in manufacturing such as faulty raw material, unskilled/careless operators, and loosening of machine settings may lead to a change in process dispersion without necessarily influencing the level of the process mean. The control charts based on the sample range, *R*, and sample standard deviation, *S*, are widely used in monitoring process dispersion.

Let $X_1, X_2, ..., X_n$ be a random sample of size *n* from a normally distributed process with process mean $\mu$ and standard deviation $\delta\sigma_0$. The sample range $R$ is defined as max{ $X_1, X_2, ..., X_n$ }- min{ $X_1, X_2, ..., X_n$ }. The sample mean $\overline{X}$ and sample standard deviation *S* are defined as

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n} \text{ and } S = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}}\text{, respectively.}$$

When the process is in control, $\delta = 1$. When the process goes out of control, $\delta > 1$ indicates an increase in $\sigma_0$ and an upper control limit $k\sigma_0$ of dispersion chart is required, and a signal is issued if $S$ or $R > k\sigma_0$. The probability of chart statistic *R* falling outside its control limit can be calculated as follows:

$$\Pr(R > k\sigma_0 | \sigma = \delta\sigma_0) = \Pr(W > \frac{k}{\delta}) = 1 - F_1(\frac{k}{\delta}),$$

where $W = R/\sigma$ denotes the relative range (both *R* and *W* depend on the same $n$ ), and $F_1(.)$ denotes its cumulative distribution function. Similarly, the probability of chart statistic *S* falling outside its control limit can be calculated as follows:

$$\Pr(S > k\sigma_0 \mid \sigma = \delta\sigma_0) = \Pr(\chi^2_{n-1} > (n-1)(\frac{k}{\delta})^2) = 1 - F_2((n-1)(\frac{k}{\delta})^2),$$

where $\chi^2_{n-1}$ denotes the chi-square random variable with *n*-1 degrees of freedom, and $F_2(.)$ denotes its cumulative distribution function.

## 3. *MD* design of dispersion control charts

Shamsuzzaman and Wu (2006) proposed $\overline{X}$ chart for minimizing the proportion of defective units. This paper will employ their model to design the *MD* dispersion control charts. The design algorithm of the *MD* dispersion chart can be described by the following optimization model.

Minimize $MD$

Subject to $ATS_0 \geq \tau$ (1)

$i \leq I$ (2)

Design variables: *n*, *h*, *k*

where *i* is the actual inspection rate. $\tau$ and *I* are minimum allowable in-control $ATS_0$ and maximum allowable inspection rate given by quality engineers, respectively. The optimization model optimizes *n*, *h*, *k* to minimize *MD* subject to the constraints on both $ATS_0$ and *i*. Among three design variables, sample size of the dispersion chart, *n*, is the only independent variable. The other two can be determined as follows.

(1) Sampling interval *h*

To satisfy constraints (2) on the inspection rate *i* and to fully utilize the available resource, it is desired to have

$i = I = n/h$.

Therefore, sampling interval will be

$h = n/I$ .

(2) Upper control limit *k*

To satisfy constraints (1) on $ATS_0$ and to make the control chart most powerful, it is desired to have

$ATS_0 = ARL \times h = \tau$ .

where *ARL* is the average run length of the dispersion control chart and the upper control limit *k* is thus determined.

After *n*, *h*, *k* have been determined, the objective function *MD* can be calculated as follows.

$$MD = \int_1^{\infty} g \cdot p(\delta) \cdot ATS(\delta) \cdot f(\delta) d\delta \quad (3)$$

where *g* is the number of units produced in a time unit. $ATS(\delta)$ and $p(\delta)$ are the average time to signal and the proportion of the defective units when the process standard deviation is $\delta\sigma_0$, respectively. $f(\delta)$ is the probability density function of $\delta$ and can be chosen by users' specification. Rayleigh distribution suggested by Shamsuzzaman and Wu (2006) is given by

$$f(\delta) = \frac{\pi\delta}{2\mu_\delta^2}\exp\left(-\frac{\pi\delta^2}{4\mu_\delta^2}\right), \quad 0 \le \delta < \infty .$$

which is characterized by a single parameter, the mean value $\mu_\delta$ of $\delta$. It is well known that a small increase in $\sigma_0$ may cause a large amount to *MD* because small increase results in large *ATS*. On the other hand, a large increase in $\sigma_0$ may also contribute a substantial amount to *MD* because a large increase makes defective proportion $p(\delta)$ great.

The calculation of *MD* formulates the process change $\delta$ as a random variable and considers the whole distribution range of $\delta$ depending on the choice of $f(\delta)$. Hence, it provides a more comprehensive evaluation of the average number of defective units produced per out-of-control case.

## 4. A numerical example

Consider a juice filling process, where the quantity of content in each bottle is the quality characteristic of interest and specified as 200 ± 0.8 cc ($USL = 200.8, LSL = 199.2$). The number of units produced per hour equals to 1000 ($g = 1000$). The quality assurance engineer specifies the minimum allowable $ATS_0$ as 370 hours ($\tau = 370$) and the maximum allowable inspection rate as five units per hour ($I = 5$). From past data, the distribution of the quantity of content can be well approximated by a normal distribution with mean 200 and standard deviation 0.2 cc ($\sigma_0 = 0.2$). The out-of-control sample values of process change are observed and estimated as 1.1 ( $\mu_\delta = 1.1$).

When a heuristic design of Shewhart *R* control chart is employed (sample size *n*=5), the resultant parameters are

*n*=5, *h*=1, *k*=5.12, *MD*=31.

When a heuristic design of Shewhart *S* control chart is employed (sample size *n*=5), the resultant parameters are

*n*=5, *h*=1, *k*=2.02, *MD*=31.

When the optimal design of Shewhart *R* control chart is employed, the resultant parameters are

*n*=8, *h*=1.6, *k*=5.31, *MD*=29.77.

When the optimal design of Shewhart *S* control chart is employed, the resultant parameters are

*n*=12, *h*=2.4, *k*=1.54, *MD*=24.87.

These four charts produce the same in-control $ATS_0$ and inspection rate *i* that are equal to the specified values. That is, all designs of control chart satisfy constraints (1) and (2). It can be seen that the optimal design of MD dispersion control charts outperform heuristic design of dispersion control charts in terms of mean number of defective units.

## References

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